TWO-LOOP GHOST-ANTIGHOST CONDENSATION FOR $SU(2)$ YANG-MILLS THEORIES IN THE MAXIMAL ABELIAN GAUGE

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Abstract

In the framework of the formalism of Cornwall et all. for composite operators I study the ghost-antighost condensation in $SU(2)$ Yang-Mills theories quantized in the Maximal Abelian Gauge and I derive analytically a condensating effective potential at two ghost loops. I find that in this approximation the one loop pairing ghost-antighost is not destroyed but no mass is generated if the ansatz for the propagator suggested by the tree level Hubbard-Stratonovich transformations is used.

1 Introduction

The ghost-antighost condensation in $SU(2)$ Yang-Mills theories quantized in the Maximal Abelian Gauge [1] was proposed by Martin Schaden [2] in 1999. The original aim was to investigate how to preserve the methods of perturbation theory when infrared divergences plague the high temperature phase of QCD [3]. In fact, the analysis of Schaden provided analytical propagators for all fields except for the Abelian photon due to a dynamically generated screening mass. Later [4, 5] this phenomenon was connected with a possible explanation of the Abelian Dominance in non Abelian gauge field theories.

This analysis was given in the mean field approximation at one loop order. In this note I will extend this analysis at two-loop order within the functional formalism of Cornwall-Jackiw-Tomboulis, which was already used to study the dynamical mass generation in the model of Cornwall-Norton [6] and the chiral symmetry breaking in quantum chromodynamics [7]. The aim of this note is to shed light on the dynamics of the ghost condensation. I will prove that the ghost-antighost propagator suggested at tree level [2, 4, 5], using Hubbard-Stratonovich transformations, is not compatible at quantum level with a dynamical mass generation.

2 $SU(2)$ Yang-Mills theories in the Maximal Abelian Gauge

I shall consider the Maximal Abelian Gauge fixed $SU(2)$ Yang-Mills action in the four dimensional continuum Minkowski space [8]

$$\begin{aligned} S &= \int d^4x \Big[-\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} \left(D^{ab}_\mu A^{b\mu} \right)^2 + \\ &+ \bar{c}^a D^{ab}_\mu D^{\mu bc} c^c - \varepsilon^{ab} \varepsilon^{cd} \bar{c}^a c^d A^b_\mu A^{c\mu} - \frac{\alpha}{4} \epsilon^{ab} \epsilon^{cd} \bar{c}^a \bar{c}^b c^c c^d \Big]. \end{aligned} \tag{1}$$

According to [9] I have chosen the diagonal generator of the gauge group $SU(2)$ as Abelian charge and I have made the following decompositions for the gluons, ghosts and antighosts fields respectively: $(A^{\mu a}, A^\mu)$, (c^a, c), $(\bar{c}^a, \bar{c})$, $a = 1, 2$ labels the off-diagonal components of the Lie-algebra valued fields.

The covariant derivative D^{ab}_μ is defined with respect to the diagonal component A_μ of the Lie Algebra valued connection

$$D^{ab}_\mu \equiv \partial_\mu \delta^{ab} - \epsilon^{ab} A_\mu. \tag{2}$$

The components of the field strength are:

$$\begin{aligned} F^a_{\mu\nu} &= D^{ab}_\mu A^b_\nu - D^{ab}_\nu A^b_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + \epsilon^{ab} A^a_\mu A^b_\nu. \end{aligned} \tag{3}$$

In the action S it has been used the partial gauge fixing condition

$$D^{ab}_\mu A^{\mu b} = 0. \tag{4}$$

The action S manifests a residual $U(1)$ gauge symmetry which can be fixed imposing for example the Landau condition

$$\partial_\mu A^\mu = 0. \tag{5}$$

In the following I will not consider the Faddeev-Popov terms related to (5) since they don't play any role.

In (1) the value of the gauge parameter α has been taken equal to the "coupling constant" of the quartic ghost-antighost interaction. In the Maximal Abelian Gauge this interaction is needed for renormalizability and appears at tree level with an arbitrary coupling in order to remove the logarithmic divergence of the full two A_μ and two A^a_μ exchange between a pair of ghost-antighost scattering [8]. This phenomenon reminds the renormalizability of scalar quantum electrodynamics [10]. In particular the model (1) depends on only one parameter, the $U(1)$ invariance is preserved at every order in perturbation theory as a consequence of the global symmetry[11]

$$c \to c + \theta \tag{6}$$

which allows for the c independence of S.

3 The effective potential and one-loop calculations

In order to investigate about the dynamical generation of the condensate

$$< 0 \mid \bar{c}^a \epsilon^{ab} c^b \mid 0 > \tag{7}$$

I will construct the Hartree-Fock approximation to the generalized effective potential [6] for the model of the previous section. This effective potential will depend only on the complete propagators of the theory $G(x,y)$ for the off-diagonal ghosts, $\Delta_a(x,y)$ and $\Delta(x,y)$ for off-diagonal and diagonal gluons respectively. A field dependence is not included, since we do not expect that any of the fields acquire a vacuum expectation value. Thus for our problem we have:

$$\begin{aligned} V(G,\Delta_a,\Delta) &= -\imath \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left[\log(S^{-1}(p)G(p)) - S^{-1}(p)G(p) + 1\right] \\ &+ \frac{\imath}{2} \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left[\log({D_a}^{-1}(p)\Delta_a(p)) - {D_a}^{-1}(p)\Delta_a(p) + 1\right] \\ &+ \frac{\imath}{2} \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}\left[\log(D^{-1}(p)\Delta(p)) - D^{-1}(p)\Delta(p) + 1\right] + V_2(G,\Delta_a,\Delta) \end{aligned} \tag{8}$$

In the previous formula all space-time and gauge indices have been suppressed. $S(p)$, $D_a(p)$ and $D(p)$ are the free propagators:

$$\begin{aligned} (D_a)^{ab}_{\mu\nu}(p) &= -\imath \frac{g^2}{p^2} \delta^{ab} \left[\eta_{\mu\nu} - \frac{(1-\alpha)p_\mu p_\nu}{p^2}\right], \\ D_{\mu\nu}(p) &= -\imath \frac{g^2}{p^2} \left[\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right], \\ S^{ab}(p) &= -\frac{\imath}{p^2}\delta^{ab}. \end{aligned} \tag{9}$$

In order to focus on the ghost-antighost condensation let us consider the approximation in which $\Delta_a(p) = D_a(p)$ and $\Delta(p) = D(p)$. It will be proved in the following that the accuracy of this approximation is under control because I work in the weak coupling regime, $g^2 << 1$. In this approximation V_2 includes the contribution of diagrams which are two-particle irreducible with respect to ghost-antighost lines only.

To compute the effective potential (8) I make the following ansatz for the ghost propagator:

$$G^{ab}(p) = -\imath \frac{p^2\delta^{ab} + \varphi(p^2)\epsilon^{ab}}{p^4 + \varphi^2(p^2)} \tag{10}$$

by defining

$$-\imath\varphi(p^2)\epsilon_{ab} = G^{-1}_{ab} - S^{-1}_{ab}. \tag{11}$$

If $\varphi(p^2)$ is constant the ansatz (10) agrees with the ghost propagator used in [2, 4, 5] by making Hubbard-Stratonovich transformations.

The behaviour of $\varphi(p^2)$ can be seen from the Dyson-Schwinger equation for the ghost propagator or equivalently from the mass gap equation [6] of the effective potential V. I will investigate about the complete system of the Dyson-Schwinger equations in the Maximal Abelian gauge in a subsequent paper. Concerning now I observe that disregarding

the tadpole terms and replacing the complete $A^\mu \bar{c}c$ vertex by the bare one (Hartree-Fock approximation):

$$\partial_\mu \bar{c^a} \epsilon^{ab} A^\mu c^b - \bar{c^a} \epsilon^{ab} A^\mu \partial_\mu c^b \tag{12}$$

the two-loop part of V is

$$V_2 = -\imath \int \frac{d^4p d^4q}{(2\pi)^8} \left[p_\rho (p_\mu + q_\mu) G^{fa}(p) \epsilon^{ac} G^{cd}(q) \epsilon^{df} D^{\mu\rho}(p-q) \right]. \tag{13}$$

The mass gap equation for (8):

$$\frac{\delta V}{\delta G} = 0 \tag{14}$$

becomes in this approximation, with the definition (11),

$$\varphi(p^2) = -4\imath g^2 \int \frac{d^4q}{(2\pi)^4} \frac{\varphi(q^2)}{q^2 (q-p)^2}, \tag{15}$$

where the propagator for A_μ in the Feynman-gauge has been used. Nevertheless for a non trivial $\varphi(p^2)$ the equation (15) is not compatible with the rest coming from the symmetric part of (10)

$$0 = g^2 \int \frac{d^4q}{(2\pi)^4} \frac{q^4}{q^4 + \varphi^2(p^2)} \frac{1}{(p-q)^2} \tag{16}$$

If I ignore this important point the result is no mass generation due to ghost condensation.

The equation (15) is similar in structure to the equation for the chirally asymmetric part of the inverse electron propagator in the Baker-Johnson-Willey approach to electrodynamics [12]. Guided by the work of these authors I ask if there is a solution to (15) whose asymptotic behaviour is:

$$\varphi(p^2) = \begin{cases} \varphi & \mid -p^2 \mid \leq \Lambda^2 \\ \varphi(-\frac{p^2}{\Lambda^2})^{-\varepsilon} & \mid -p^2 \mid \gg \Lambda^2 \end{cases} \tag{17}$$

in which Λ is taken as a fixed massive parameter. Of course $\varphi(p^2)$ must be a continuous function and one should specify the transition between the high energy and the low energy behaviour. However various reasonable transition behaviours make only a small difference in the numerical coefficient of the final result of the effective potential [6].

The integral equation (15) is equivalent to the following differential equation:

$$\frac{d}{dx}\left(x^2 \frac{d}{dx} \varphi(x) \right) = -\frac{4g^2}{16\pi^2} \varphi(x). \tag{18}$$

If I put the ansatz (17) I obtain, for $g^2 << 1$, the solution

$$\varepsilon = \frac{4g^2}{16\pi^2} + O(g^2). \tag{19}$$

Because ε is small, the ansatz (17) is a good approximation also in the infrared domain [6, 13]. However in the following $\varepsilon \to 0$, playing the role of a regulator, therefore in any gauge I will assume an order of magnitude given by (19).

I would like to stress that $\varphi(p^2)\epsilon_{ab}$ is, in my notation, the antisymmetric part of the propagator G but φ is p^2-independent and plays the role of some suitably regularized value of $< 0 \mid \bar{c}^a \epsilon^{ab} c^b \mid 0 >$.

The one-loop contribution to (8) up to φ-independent terms is obtained from (9) and (10):

$$V_1(\varphi) = -\imath \int \frac{d^4p}{(2\pi)^4}\left[\log\left(1 - \frac{\varphi^2(p^2)}{p^4 + \varphi^2(p^2)}\right) + \frac{2\varphi^2(p^2)}{p^4+\varphi^2(p^2)}\right]. \tag{20}$$

This expression takes the following form in the Euclidean region:

$$V_1(\varphi) = -\frac{1}{16\pi^2}\int_0^{+\infty} dx\ x\left[\log\left(1 - \frac{\varphi^2(x)}{x^2 + \varphi^2(x)}\right) + \frac{2\varphi^2(x)}{x^2+\varphi^2(x)}\right]. \tag{21}$$

The evaluation proceeds by inserting (17) into (21) and keeping only terms that are proportional to inverse power of ε as well of zero-order in ε. In practice I set ε to zero everywhere as long as no divergence arises; if $\varepsilon = 0$ is not allowed (17) is used. Therefore

$$\begin{aligned} V_1(\varphi) = & -\frac{1}{16\pi^2}\int_0^{\Lambda^2} dx\ x\left[\log\left(1 - \frac{\varphi^2}{x^2 + \varphi^2}\right) + \frac{2\varphi^2}{x^2+\varphi^2}\right] \\ & -\frac{1}{16\pi^2}\int_{\Lambda^2}^{+\infty} dx\ x\left[\log\left(1 - \frac{\varphi^2\cdot(\frac{x}{\Lambda^2})^{-2\varepsilon}}{x^2 + \varphi^2\cdot(\frac{x}{\Lambda^2})^{-2\varepsilon}}\right) + \frac{2\varphi^2\cdot(\frac{x}{\Lambda^2})^{-2\varepsilon}}{x^2+\varphi^2\cdot(\frac{x}{\Lambda^2})^{-2\varepsilon}}\right]. \end{aligned} \tag{22}$$

Performing the Laurent expansion around $\varepsilon = 0$ we get the result:

$$V_1 = \frac{\varphi^2}{32\pi^2} - \frac{\varphi^2}{32\pi^2\varepsilon} + \frac{\varphi^2}{32\pi^2}\log\left(\frac{\varphi^2}{\Lambda^4}\right) \tag{23}$$

It agrees with the computed result in the $\overline{\mathrm{MS}}$ scheme [4].

4 Contributions of Two-Loop Diagrams

Now let me consider the two-loop contribution to the effective potential. I am looking for connected, two-particle irreducible graphs of order $\hbar^2$ in the expression:

$$\begin{aligned} \imath\hbar < 0 \mid \quad & T\exp\left\{-\imath\hbar\int d^4x\left[\partial_\mu\bar{c}^a\epsilon^{ab}A^\mu c^b - \bar{c}^a\epsilon^{ab}A^\mu\partial_\mu c^b + \frac{\alpha}{2}\epsilon^{ab}\epsilon^{cd}\bar{c}^a c^b\bar{c}^c c^d\right.\right. \\ & \left.\left. -\bar{c}^a c^a A_\mu A^\mu - \epsilon^{ab}\epsilon^{cd}\bar{c}^a c^d A^b_\mu A^{\mu c}\right]\right\} \mid 0 >, \end{aligned} \tag{24}$$

the parameter $\hbar$ has been introduced in order to count loops, but it will be put equal to one at the end of the calculation. Upon scaling the fields in (24) like $\psi \to \hbar^{1/2}\psi$, expanding the exponential to the relevant order and applying Wick's theorem, I am left with four integrals. Let me consider the first one:

$$\begin{aligned} \frac{I_1}{\hbar^2} \quad = \quad & g^2\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\left\{\frac{p_\rho(p_\mu+q_\mu)}{(p-q)^2}\left(\eta^{\mu\rho} - \frac{(p^\mu-q^\mu)(p^\rho-q^\rho)}{(p-q)^2}\right)\right. \\ & \left.\left[-\frac{2p^2q^2}{(p^4+\varphi^2(p^2))(q^4+\varphi^2(q^2))} + \frac{2\varphi(p^2)\varphi(q^2)}{(p^4+\varphi^2(p^2))(q^4+\varphi^2(q^2))}\right]\right\}. \end{aligned} \tag{25}$$

After making some standard integration on the angles [15], I get in the Euclidean region

$$\begin{aligned} \frac{I_1}{\hbar^2} \quad = \quad & \frac{3g^2}{256\pi^4}\int_0^{+\infty} dxdy\left\{\frac{xy}{(x^2+\varphi^2(x))(y^2+\varphi^2(y))} - \frac{\varphi(x)\varphi(y)}{(x^2+\varphi^2(x))(y^2+\varphi^2(y))}\right\}\times \\ & \times\left[y^2\theta(x-y) + x^2\theta(y-x)\right]. \end{aligned} \tag{26}$$

Using the expression given in (17) I obtain the following decomposition

$$\begin{aligned}
\frac{I_1}{\hbar^2} = \frac{3g^2}{128\pi^4} \quad \times \quad & \left[\int_0^{\Lambda^2} dy \frac{y^3}{y^2+\varphi^2} \int_y^{\Lambda^2} dx \frac{x}{x^2+\varphi^2} - \int_0^{\Lambda^2} dy \frac{y^2}{y^2+\varphi^2} \int_y^{\Lambda^2} dx \frac{\varphi^2}{x^2+\varphi^2}\right. \\
+ \quad & \int_0^{\Lambda^2} dy \frac{y^3}{y^2+\varphi^2} \int_y^{\Lambda^2} dx \frac{x}{x^2+\varphi^2\left(\frac{x}{\Lambda^2}\right)^{-2\varepsilon}} \\
- \quad & \int_0^{\Lambda^2} dy \frac{y^2}{y^2+\varphi^2} \int_{\Lambda^2}^{+\infty} dx \frac{\varphi^2\left(\frac{x}{\Lambda^2}\right)^{-\varepsilon}}{x^2+\varphi^2\left(\frac{x}{\Lambda^2}\right)^{-2\varepsilon}} \\
+ \quad & \int_{\Lambda^2}^{+\infty} dy \frac{y^3}{y^2+\varphi^2\left(\frac{y}{\Lambda^2}\right)^{-2\varepsilon}} \int_y^{+\infty} dx \frac{x}{x^2+\varphi^2\left(\frac{x}{\Lambda^2}\right)^{-2\varepsilon}} \\
- \quad & \left.\int_{\Lambda^2}^{+\infty} dy \frac{y^2\left(\frac{y}{\Lambda^2}\right)^{-\varepsilon}}{y^2+\varphi^2\left(\frac{y}{\Lambda^2}\right)^{-2\varepsilon}} \int_y^{+\infty} dx \frac{\varphi^2\left(\frac{x}{\Lambda^2}\right)^{-\varepsilon}}{x^2+\varphi^2\left(\frac{x}{\Lambda^2}\right)^{-2\varepsilon}}\right]. \qquad (27)
\end{aligned}$$

After making analytical continuation [16] and Laurent expansion of (27) around $\varepsilon = 0$ I get [17]:

$$\frac{I_1}{\hbar^2} = \frac{3g^2}{512\pi^4\varepsilon}\varphi^2 + \frac{3g^2\varphi^2}{256\pi^4}\left(-\frac{\pi^2}{6}+\frac{1}{2}\right) - \frac{3g^2}{512\pi^4}\varphi^2\log\left(\frac{\varphi^2}{\Lambda^4}\right). \qquad (28)$$

In the appendix I will give more details about how I performed the integrals of (27). Now let me consider the second integral coming from the expansion of (24):

$$\frac{I_2}{\hbar^2} = -\alpha\left\{\left[\int \frac{d^4p}{(2\pi)^4}\frac{\varphi(p^2)}{p^4+\varphi^2(p^2)}\right]^2 + \left[\int \frac{d^4p}{(2\pi)^4}\frac{p^2}{p^4+\varphi^2(p^2)}\right]^2\right\}. \qquad (29)$$

Substituting the expression (17) in (24) I get for the first term after the usual change of variables $p^0 \longrightarrow \imath p^0$, analytical continuation [16] around $\varepsilon = 0$:

$$\frac{I_2}{\hbar^2} = \frac{\alpha\varphi^2}{256\pi^4}\left(-\frac{1}{2}\log\left(\frac{\varphi^2}{\Lambda^4}\right)+\frac{1}{\varepsilon}\right)^2. \qquad (30)$$

and for the second term

$$\int \frac{d^4p}{(2\pi)^4}\frac{p^2}{p^4+\varphi^2(p^2)} = O(\varepsilon), \qquad (31)$$

It will proved in the appendix.

Finally it is easy to see that the sum of the last two integrals that can be extracted from (24) is:

$$\frac{I_3+I_4}{\hbar^2} = 2\int \frac{d^4p}{(2\pi)^4}\frac{p^2}{p^4+\varphi^2(p^2)}\int \frac{d^4q}{(2\pi)^4}\frac{\alpha}{q^2} \qquad (32)$$

and it is $O(\varepsilon)$ due to (31).

By using the same method and defining for massive off-diagonal gluons the following propagator:

$$(\Delta_a)^{ab}_{\mu\nu}(p) = -\imath\frac{g^2}{p^2-M^2(p^2)}\delta^{ab}\left[\eta_{\mu\nu} - \frac{(1-\alpha)p_\mu p_\nu}{p^2-M^2(p^2)}\right] \qquad (33)$$

with

$$M^2(p^2) = \begin{cases} M^2 & |-p^2| \leq \Lambda^2 \\ M^2\cdot(-\frac{p^2}{\Lambda^2})^{-2\varepsilon} & |-p^2| \gg \Lambda^2 \end{cases} \qquad (34)$$

it is easy to see that the vertex $\bar{c}cAA$ will provide a $O(\varepsilon)$ contribution to the effective potential V, which must be disregarded for $g^2 << 1$. The main point of this paper is here. If one uses the ansatz (10), the effective potential doesn't possess, at the lowest order in the weak coupling regime, the necessary mixing term between M and φ for the generation of a mass for off-diagonal gluons related to the ghost-antighost condensate. That is because the symmetric part of (10) doesn't satisfy the Dyson-Schwinger equations.

Moreover it is possible to say that due to (31) the approximation $\Delta_a = D_a$ is compatible with the weak coupling regime. Since the propagator Δ is supposed to coincide with the normal perturbative solution because no symmetry-breaking effects are expected, the weak coupling regime controls also the approximation $\Delta = D$.

5 Effective potential and the ghost condensate

Collecting the results found in the previous section and keeping only terms that are proportional to inverse powers of the coupling g (these come from inverse powers of ε) as well as terms of zeroth order in ε and coupling I get the two-loop effective potential

$$\begin{aligned} V(\varphi) &= \frac{\varphi^2}{32\pi^2}\left(1-\frac{1}{\varepsilon}\right)+\frac{1}{32\pi^2}\varphi^2\log\left(\frac{\varphi^2}{\Lambda^4}\right) \\ &+ \frac{\alpha\varphi^2}{256\pi^4}\left(\frac{1}{2}\log\left(\frac{\varphi^2}{\Lambda^4}\right)-\frac{1}{\varepsilon}\right)^2, \end{aligned} \tag{35}$$

where terms divergent at $\varepsilon = 0$ but multiplied by higher powers of the coupling constant have been dropped.

In the weak coupling regime the effective potential is independent on the gauge parameter ζ of the $U(1)$ symmetry. In fact it is easy to check, using the results of the previous section, that in a general covariant gauge one should add to (35)

$$\frac{\zeta g^2\varphi^2}{256\pi^4}\left(\frac{1}{2}\log\left(\frac{\varphi^2}{\Lambda^4}\right)-\frac{1}{\varepsilon}\right)^2 \tag{36}$$

which is negligible compared to the term proportional to α if $\alpha >> g^2$.

Although (35) is not the end of the story, it is worth to remark that the effective potential $V(\varphi)$ must be bounded from below therefore:

$$\alpha > 0 \tag{37}$$

which is equivalent to state the concavity of $V(\varphi)$[18]. Moreover since this potential manifests a nontrivial absolute minimum if

$$\alpha > -\frac{\varepsilon}{4} \tag{38}$$

and since we work for $\varepsilon \to 0$ the inequality (38) is satisfied if $V(\varphi)$ is bounded from below.The absolute minimum of our effective potential is found to be at:

$$\log\left(\frac{\varphi^2}{\Lambda^4}\right) = \frac{2}{\varepsilon} - 1 - \frac{16\pi^2}{\alpha}. \tag{39}$$

I observe that the quartic ghost-antighost interaction seems to play a crucial role in this mechanism of condensation. This interaction seems to affect the effective potential much

more than the cubic vertex $\bar{c}cA$ which only perturbs the one-loop result. It is worth to remark that α positive could be related to a sort of "ghost-attraction", but unfortunately I don't have any general argument to state the positivity of

$$-\frac{\alpha}{4}\epsilon^{ab}\epsilon^{cd}\bar{c}^a\bar{c}^b c^c c^d = \alpha\bar{c}^1 c^1\bar{c}^2 c^2 \tag{40}$$

when the usual assignments of hermiticity [19]

$$\begin{aligned} c^\dagger &= c \\ \bar{c}^\dagger &= -\bar{c} \end{aligned} \tag{41}$$

are given.

The contributions to the effective potential proportional to α are dominated by the term

$$\frac{\alpha\varphi^2}{1024\pi^4}\log^2\left(\frac{\varphi^2}{\Lambda^4}\right) \tag{42}$$

which is clearly a symmetry restoring term. Nevertheless if

$$\alpha \approx 16\pi^2\varepsilon \tag{43}$$

the absolute minima of V_1 and V are on the same value and easily it is possible to see:

$$\left(\frac{V}{\varphi^2} - \frac{V_1}{\varphi^2}\right)_{\min} = O(g^2). \tag{44}$$

Therefore for $\alpha \sim O(g^2)$ the two-loop contribution corresponds to a small perturbation of the one-loop result.

6 Discussion

I have derived a two-loop ghost-antighost condensating effective potential in the weak coupling regime using an ansatz found at tree level, but not efficient at quantum level. The consequence of this wrong ansatz has been that in the off-diagonal gluon propagator no mass or infrared cut-off is generated as claimed in [2, 4]. In order to improve this study it becomes mandatory to know more about the complete propagators of the theory, for example from the complicated system of the Dyson-Schwinger equations. The complete propagators are expected to show a richer structure than in (10) due to the dependence on the most general BRST invariant[20] condensate of dimension two

$$< 0|A^a_\mu A^{\mu a} + \alpha\bar{c}^a c^a|0 > . \tag{45}$$

This condensate for Yang-Mills theories in the Maximal Abelian Gauge is now under investigations [21]. For its computation could be crucial the residual $U(1)$ gauge invariance of the theory after the partial gauge fixing condition (4). In fact taking $\alpha = -1$ and calling ξ the coupling constant of the self interaction between ghosts the action can be written:

$$\begin{aligned} S &= \int d^4x\left[-\frac{1}{4g^2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2g^2}A_\nu{}^a D^{ab}_\mu D^\mu{}_{bc}A^{\nu c} - \frac{1}{2g^2}(\epsilon^{ab}A_\nu{}^a A^b_\mu)^2\right. \\ &\left. +\bar{c}^a D^{ab}_\mu D^{\mu bc}c^c + \frac{\xi}{2}(\bar{c}^a c^a)^2 - \varepsilon^{ab}\varepsilon^{cd}\bar{c}^a c^d A_\mu{}^b A^{c\mu}\right]. \end{aligned} \tag{46}$$

This action represents a sort of scalar electrodynamics of charged off-diagonal gluons and the off-diagonal ghosts and antighosts fields interacting each other by usual quartic scalar terms. These classes of models, constraints by the vanishing of the vacuum expectation value of every charged scalar fields, provided a stable vacuum due the pair condensates of the charged scalar fields [22]. Using these results the condensate (45) could be evaluated providing a gauge invariant mass generation for continuum Yang-Mills theories.

A Integrals

In this appendix I will give more details about the calculations of two integrals met in section 3.

The first integral

$$J_1 = \int_{\Lambda^2}^{+\infty} dy \frac{y^3}{y^2 + \varphi^2 \left(\frac{y}{\Lambda^2}\right)^{-2\varepsilon}} \int_y^{+\infty} dx \frac{x}{x^2 + \varphi^2 \left(\frac{x}{\Lambda^2}\right)^{-2\varepsilon}} \tag{47}$$

is easily shown to be equal to :

$$\Lambda^4 \int_1^{+\infty} dy \frac{y^3}{y^2 + f^2 y^{-2\varepsilon}} \int_y^{+\infty} dx \frac{x}{x^2 + f^2 x^{-2\varepsilon}}. \tag{48}$$

with $f^2 = \frac{\varphi^2}{\Lambda^2}$. Since[17]

$$\int_y^{+\infty} dx \frac{x}{x^2 + f^2 x^{-2\varepsilon}} = \frac{1}{2(1+\varepsilon)} \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right) \tag{49}$$

our integral becomes in its convergence region [16]

$$J_1 = \frac{\Lambda^4}{2(1+\varepsilon)} \int_1^{+\infty} dy \frac{y^{3+2\varepsilon}}{y^{2+2\varepsilon} + f^2} \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right) \tag{50}$$

I adopt the following trick

$$\begin{aligned} J_1 &= \frac{\Lambda^4}{2(1+\varepsilon)} \int_1^{+\infty} dy \left[\left(\frac{y^{3+2\varepsilon}}{y^{2+2\varepsilon} + f^2} - y + \frac{f^2}{y}\right) \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right)\right. \\ &\quad \left. + \left(y - \frac{f^2}{y}\right) \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right)\right]. \end{aligned} \tag{51}$$

But

$$\begin{aligned} &\int_1^{+\infty} dy \frac{f^4 - f^2 y^2 + f^2 y^{2+2\varepsilon}}{y(f^2 + y^{2+2\varepsilon})} \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right) = \\ &\int_1^{+\infty} dy \frac{f^4}{y(f^2 + y^2)} \log\left(1 + \frac{y^2}{f^2}\right) + O(\varepsilon) = \\ &\frac{f^2}{12}\left[3\log^2\left(1 + \frac{1}{f^2}\right) + \pi^2 + 6\mathrm{Li}_2\left(-\frac{1}{f^2}\right)\right] + O(\varepsilon) \end{aligned} \tag{52}$$

where in the last equality $\mathrm{Li}_2(x)$ is the dilogarithm function, with the property:

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log x \log(1-x). \tag{53}$$

and it has been used the change to the variable $z = \log\left(1 + \frac{y}{f^2}\right)$.

Moreover[17]

$$\int_1^{+\infty} dy \qquad y \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right) = \\ \frac{f^2}{2\varepsilon} + \frac{1}{2}\left[(1+f^2)\left(1 - \log\left(1 + \frac{1}{f^2}\right)\right) - f^2 \log f^2\right] + O(\varepsilon). \tag{54}$$

Finally[17]

$$\int_1^{+\infty} \frac{dy}{y} \log\left(1 + \frac{y^{2+2\varepsilon}}{f^2}\right) = \frac{f^2}{2}\mathrm{Li}_2\left(-\frac{1}{f^2}\right) + O(\varepsilon). \tag{55}$$

The final result is

$$\begin{aligned} J_1 &= \frac{\varphi^2}{2\varepsilon} + \frac{\Lambda^4}{2} - \frac{\Lambda^4}{2}\log\left(1 + \frac{1}{f^2}\right) + \frac{\varphi^2}{4}\log^2\left(1 + \frac{1}{f^2}\right) \\ &\quad - \frac{\varphi^2}{2}\log\left(1 + \frac{1}{f^2}\right) + \varphi^2\left(\frac{\pi^2}{12} + \frac{1}{2}\right) + O(\varepsilon) \end{aligned} \tag{56}$$

Now I will prove the (31):

$$\int \frac{d^4p}{(2\pi)^4} \frac{p^2}{p^4 + \varphi^2(p^2)} = O(\varepsilon). \tag{57}$$

Using hyperspherical Euclidean coordinates the integral becomes proportional to

$$\int_0^{\Lambda^2} dx \frac{x^2}{x^2 + \varphi^2} + \int_{\Lambda^2}^{+\infty} dx \frac{x^2}{x^2 + \varphi^2 x^{-2\varepsilon}}. \tag{58}$$

But [17]

$$\int_{\Lambda^2}^{+\infty} dx \frac{x^2}{x^2 + \frac{\varphi^2}{\Lambda^2} x^{-2\varepsilon}} = -\frac{\Lambda^2}{(3+2\varepsilon)\varphi^2} \; {}_2F_1\left(1, \frac{3+2\varepsilon}{1+2\varepsilon}, \frac{4+4\varepsilon}{1+2\varepsilon}, -\frac{\Lambda^2}{\varphi^2}\right) \tag{59}$$

$$\text{If} \quad Re(\varepsilon) < -\frac{3}{2}.$$

Since (59) can be prolonged [16] at $\varepsilon = 0$, the Laurent expansion of (57) is there:

$$\Lambda^2\left(-1 + \frac{\varphi}{\Lambda^2}\arctan\frac{\Lambda^2}{\varphi} + O(\varepsilon)\right) = -\int_0^{\Lambda^2} dx \frac{x^2}{x^2 + \varphi^2} + O(\varepsilon) \tag{60}$$

and I get the result (57).

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